University of Thessaly

Department of Computer and Communication Engineering

Technical Report

Some results on diagonally dominant matrices with positive diagonal elements

Definition. A $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is called *row diagonally dominant* if $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$, $\forall i = 1, \ldots, n$. It is called *column diagonally dominant* if $|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|$, $\forall j = 1, \ldots, n$.

Lemma 1. If $\mathbf{A} = [a_{ij}]$ is a $n \times n$ row (resp., column) diagonally dominant matrix with positive diagonal elements, and $\mathbf{D} = [d_i]$, $\mathbf{C} = [c_i]$ are $n \times n$ positive and nonnegative diagonal matrices respectively (i.e. $d_i > 0$ and $c_i \geq 0$, $i = 1, \ldots, n$), then the matrix $\mathbf{B} = \mathbf{C} + \mathbf{DA}$ (resp., $\mathbf{B} = \mathbf{C} + \mathbf{AD}$) is row (resp., column) diagonally dominant with positive diagonal elements.

Proof. It is easily observed that, for a diagonal matrix **D**, the *ij*th element of the matrix product **DA** equals $d_i a_{ij}$ (i.e. the effect of pre-multiplying a matrix **A** by a diagonal matrix **D** is simply to multiply each element of the *i*th row of **A** by the *i*th diagonal element of **D**). Similarly, the *ij*th element of the product **AD** equals $d_j a_{ij}$ (i.e. the effect of post-multiplying a matrix **A** by a diagonal matrix **D** is to multiply each element of the *j*th column of **A** by the *j*th diagonal element of **D**). Now, if **A** is row diagonally dominant with positive diagonal elements, then for the matrix $\mathbf{B} = \mathbf{C} + \mathbf{DA}$ it holds $b_{ii} = c_i + d_i a_{ii} > 0$ ($i = 1, \ldots, n$) and also:

$$b_{ii} = c_i + d_i a_{ii} > c_i + d_i \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |d_i a_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}|, \ \forall i = 1, \ldots, n$$

i.e. **B** is row diagonally dominant with positive diagonal elements.

If **A** is column diagonally dominant with positive diagonal elements, then for the matrix $\mathbf{B} = \mathbf{C} + \mathbf{AD}$ it holds $b_{jj} = c_j + d_j a_{jj} > 0$ ($j = 1, \ldots, n$) and also:

$$b_{jj} = c_j + d_j a_{jj} > c_j + d_j \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |d_j a_{ij}| = \sum_{\substack{i=1 \\ i \neq j}}^{n} |b_{ij}|, \ \forall j = 1, \ldots, n$$

i.e. **B** is column diagonally dominant with positive diagonal elements. **Q.E.D.**

Lemma 2. If $\mathbf{A} = [a_{ij}]$ is a $n \times n$ matrix with positive diagonal elements which satisfies $\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| < 1$ (resp., $\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| < 1$), then the matrix $\mathbf{B} = \mathbf{I} - \mathbf{A}$ is row (resp., column) diagonally dominant with positive diagonal elements.

Proof. If $\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| < 1$, then $\sum_{j=1}^{n} |a_{ij}| < 1$, $\forall i = 1, \ldots, n$, or (considering also that $a_{ii} > 0$, $i = 1, \ldots, n$):

$$0 < \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}| < 1 - a_{ii} = b_{ii}, \ \forall i = 1, \ldots, n$$

i.e. $\mathbf{B} = \mathbf{I} - \mathbf{A}$ is row diagonally dominant with positive diagonal elements.

Likewise, if $\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| < 1$, then $\sum_{i=1}^{n} |a_{ij}| < 1$, $\forall j = 1, \ldots, n$, or (considering also that $a_{jj} > 0$, $j = 1, \ldots, n$):

$$0 < \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| = \sum_{\substack{i=1 \\ i \neq j}}^{n} |b_{ij}| < 1 - a_{jj} = b_{jj}, \ \forall j = 1, \ldots, n$$

i.e. $\mathbf{B} = \mathbf{I} - \mathbf{A}$ is column diagonally dominant with positive diagonal elements. **Q.E.D.**

Theorem 1. If $\mathbf{A} = [a_{ij}]$ is a $n \times n$ row diagonally dominant matrix with positive diagonal elements then $\left\| \mathbf{A}^{-1} \right\|_{\infty} \leq \left[\min_{1 \leq i \leq n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right) \right]^{-1}$. If $\mathbf{A}$ is column diagonally dominant with positive diagonal elements then $\left\| \mathbf{A}^{-1} \right\|_{1} \leq \left[\min_{1 \leq j \leq n} \left(a_{jj} - \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \right) \right]^{-1}$.

Proof. For every induced matrix norm it is:

$$\left\| \mathbf{A}^{-1} \right\| = \sup_{\substack{\mathbf{x} \in \Re^n \\ \mathbf{x} \neq \mathbf{0}}} \frac{\left\| \mathbf{A}^{-1}\mathbf{x} \right\|}{\|\mathbf{x}\|} = \sup_{\substack{\mathbf{x} \in \Re^n \\ \mathbf{x} \neq \mathbf{0}}} \frac{\left\| \mathbf{A}^{-1}\mathbf{x} \right\|}{\left\| \mathbf{A}\mathbf{A}^{-1}\mathbf{x} \right\|} = \sup_{\substack{\mathbf{y} \in \Re^n \\ \mathbf{y} \neq \mathbf{0}}} \frac{\|\mathbf{y}\|}{\|\mathbf{A}\mathbf{y}\|} = \left[\inf_{\substack{\mathbf{y} \in \Re^n \\ \mathbf{y} \neq \mathbf{0}}} \frac{\|\mathbf{A}\mathbf{y}\|}{\|\mathbf{y}\|} \right]^{-1}$$

(provided, of course, that $\mathbf{A}$ is nonsingular). Now, if $\mathbf{A}$ is row diagonally dominant with positive diagonal elements (in which case $\mathbf{A}^{-1}$ always exists [1]), then in order to show that $\left\| \mathbf{A}^{-1} \right\|_{\infty} \leq \left[\min_{1 \leq i \leq n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right) \right]^{-1}$ we just need to show that $\frac{\|\mathbf{A}\mathbf{y}\|_{\infty}}{\|\mathbf{y}\|_{\infty}} \geq \min_{1 \leq i \leq n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right) > 0$, $\forall \mathbf{y} \in \Re^n$. Assume that for some arbitrary vector $\mathbf{y} \in \Re^n$ it is $|y_k| = \|\mathbf{y}\|_{\infty} = \max_{1 \leq i \leq n} |y_i|$. Then we have:

$$\frac{\|\mathbf{A}\mathbf{y}\|_{\infty}}{\|\mathbf{y}\|_{\infty}} = \frac{\max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij} y_j \right|}{|y_k|} \geq \frac{\left| \sum_{j=1}^{n} a_{kj} y_j \right|}{|y_k|} \geq \frac{|a_{kk} y_k| - \left| \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} y_j \right|}{|y_k|} \geq \frac{|a_{kk} y_k| - \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| |y_j|}{|y_k|}$$

$$\geq \frac{|a_{kk} y_k| - \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}||y_k|}{|y_k|} = a_{kk} - \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| \geq \min_{1\leq i\leq n} \left(a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right) > 0$$

Similarly, if **A** is column diagonally dominant with positive diagonal elements we need to show that $\frac{\|\mathbf{Ay}\|_1}{\|\mathbf{y}\|_1} \geq \min_{1\leq j\leq n} \left(a_{jj} - \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \right) > 0$, $\forall \mathbf{y} \in \Re^n$. For some arbitrary vector $\mathbf{y} \in \Re^n$ we have:

$$\frac{\|\mathbf{Ay}\|_1}{\|\mathbf{y}\|_1} = \frac{\sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} y_j \right|}{\sum_{i=1}^{n} |y_i|} \geq \frac{\sum_{i=1}^{n} \left(|a_{ii} y_i| - \left| \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} y_j \right| \right)}{\sum_{i=1}^{n} |y_i|} \geq \frac{\sum_{i=1}^{n} \left(|a_{ii} y_i| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}||y_j| \right)}{\sum_{i=1}^{n} |y_i|}$$

$$= \frac{\sum_{j=1}^{n} \left(|a_{jj} y_j| - \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}||y_j| \right)}{\sum_{j=1}^{n} |y_j|} = \frac{\sum_{j=1}^{n} \left(|y_j| \left(a_{jj} - \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \right) \right)}{\sum_{j=1}^{n} |y_j|} \geq \frac{\left(\sum_{j=1}^{n} |y_j| \right) \min_{1\leq j\leq n} \left(a_{jj} - \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \right)}{\sum_{j=1}^{n} |y_j|}$$

$$= \min_{1\leq j\leq n} \left(a_{jj} - \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \right) > 0$$ **Q.E.D.**

Theorem 2. Let $\mathbf{A} = [a_{ij}]$ be a $n \times n$ – row or column – diagonally dominant matrix with positive diagonal elements. If λ is an eigenvalue of **A** then $\operatorname{Re}\lambda > 0$.

Proof. This is an immediate consequence of the Gershgorin circle theorem [1], by which every eigenvalue λ_k, $k = 1,...,n$ of a square matrix **A** is located in one of the n disks in the complex plane defined by $\left\{ z : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\}$, $i = 1,...,n$ (i.e. the n disks centered at a_{ii} and having radius $\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$, $i = 1,...,n$). Obviously, if the matrix **A**

is row diagonally dominant with positive diagonal elements, then all Gershgorin disks lie entirely in the positive real semi-plane and thus all eigenvalues of **A** have positive real parts, i.e. $\mathrm{Re}\,\lambda_k > 0$, $\forall k = 1,...,n$. Since the Gershgorin circle theorem can be restated for the set of disks $\left\{ z : |z - a_{jj}| \le \sum_{\substack{i=1 \\ i \ne j}}^{n} |a_{ij}| \right\}$, $j = 1,...,n$ (by applying it to $\mathbf{A}^T$ and because $\mathbf{A}, \mathbf{A}^T$ have the same eigenvalues [2]), it holds again $\mathrm{Re}\,\lambda_k > 0$, $\forall k = 1,...,n$ for the case where **A** is column diagonally dominant with positive diagonal elements.

There is an alternative way of proving the theorem. Suppose, to derive a contradiction, that there exists an eigenvalue λ of **A** which has $\mathrm{Re}\,\lambda \le 0$. Then, if **A** is row diagonally dominant with positive diagonal elements it would be $|\lambda - a_{ii}| = \sqrt{\left(a_{ii} + |\mathrm{Re}\,\lambda|\right)^2 + \left(\mathrm{Im}\,\lambda\right)^2} > a_{ii} > \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$, which means that the matrix $\lambda\mathbf{I} - \mathbf{A}$ is also row diagonally dominant (generally, with complex diagonal elements). However such a matrix is always nonsingular [1], i.e. $\det(\lambda\mathbf{I} - \mathbf{A}) \ne 0$, which contradicts our initial hypothesis that λ is an eigenvalue of **A** (a similar proof can be derived for the case of **A** being column diagonally dominant with positive diagonal elements). **Q.E.D.**

Theorem 3. If $\mathbf{A} = [a_{ij}]$ is a $n \times n$ – row or column – diagonally dominant matrix with positive diagonal elements then $\mathbf{A}^{-1}$ has only positive diagonal elements.

Proof. Let $\mathbf{A}^{-1} = [\alpha_{ij}]$. For the diagonal elements α_{ii} of $\mathbf{A}^{-1}$ it holds:

$$\alpha_{ii} = (-1)^{i+i} \frac{\det(\mathbf{A}_{ii})}{\det(\mathbf{A})} = \frac{\det(\mathbf{A}_{ii})}{\det(\mathbf{A})},\ i = 1,...,n$$

where $\mathbf{A}_{ii}$ is the (principal) submatrix of **A** obtained by striking out the *i*th row and the *i*th column. If **A** is – row or column – diagonally dominant with positive diagonal elements, then so is every principal submatrix $\mathbf{A}_{ii}$, $i = 1,...,n$, as is easily verified. Thus, by Theorem 2 all eigenvalues of **A** as well as of any principal submatrix $\mathbf{A}_{ii}$,

$i=1,\ldots,n$ have positive real parts. Let μ_k, $k=1,\ldots,n_r$ and $\gamma_k, \bar{\gamma}_k$, $k=1,\ldots,n_c$ be the – not necessarily distinct – real and complex eigenvalues of $\mathbf{A}$ respectively (the latter occurring in conjugate pairs), where $n_r + 2n_c = n$. Let also $\mu_k^{(i)}$, $k=1,\ldots,n_r^{(i)}$ and $\gamma_k^{(i)}, \bar{\gamma}_k^{(i)}$, $k=1,\ldots,n_c^{(i)}$ denote the real and complex eigenvalues of the principal submatrix $\mathbf{A}_{ii}$, $i=1,\ldots,n$, where $n_r^{(i)} + 2n_c^{(i)} = n-1$ for every $i=1,\ldots,n$. Since the determinant of any square matrix is equal to the product of its eigenvalues [2], we arrive at the desired result:

$$\alpha_{ii} = \frac{\det(\mathbf{A}_{ii})}{\det(\mathbf{A})} = \frac{\prod_{k=1}^{n_r^{(i)}} \mu_k^{(i)} \prod_{k=1}^{n_c^{(i)}} \left[\left(\operatorname{Re}\gamma_k^{(i)}\right)^2 + \left(\operatorname{Im}\gamma_k^{(i)}\right)^2 \right]}{\prod_{k=1}^{n_r} \mu_k \prod_{k=1}^{n_c} \left[\left(\operatorname{Re}\gamma_k\right)^2 + \left(\operatorname{Im}\gamma_k\right)^2 \right]} > 0, \; i=1,\ldots,n \qquad \textbf{Q.E.D.}$$

Theorem 4. If $\mathbf{A}=[a_{ij}]$ is a $n \times n$ row (resp., column) diagonally dominant matrix with positive diagonal elements then the matrix $\mathbf{I}-(\mathbf{I}+\mathbf{A})^{-1}$ $= (\mathbf{I}+\mathbf{A})^{-1}\mathbf{A} = (\mathbf{A}^{-1}+\mathbf{I})^{-1}$ is also row (resp., column) diagonally dominant with positive diagonal elements.

Proof. If $\mathbf{A}$ is row diagonally dominant with positive diagonal elements then clearly the same holds for $\mathbf{I}+\mathbf{A}$. This implies that the inverse $(\mathbf{I}+\mathbf{A})^{-1}$ has only positive diagonal elements (due to Theorem 3) and also satisfies (on account of Theorem 1):

$$\left\| (\mathbf{I}+\mathbf{A})^{-1} \right\|_\infty < \left[\min_{1 \le i \le n} \left(1 + a_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right) \right]^{-1} < 1$$

Thus Lemma 2 is applicable and the matrix $\mathbf{I}-(\mathbf{I}+\mathbf{A})^{-1}$ is row diagonally dominant with positive diagonal elements.

Likewise, if $\mathbf{A}$ is column diagonally dominant with positive diagonal elements then so is $\mathbf{I}+\mathbf{A}$, whose inverse $(\mathbf{I}+\mathbf{A})^{-1}$ has only positive diagonal elements and satisfies:

$$\left\| (\mathbf{I}+\mathbf{A})^{-1} \right\|_1 < \left[\min_{1 \le j \le n} \left(1 + a_{jj} - \sum_{\substack{i=1 \\ i \ne j}}^{n} |a_{ij}| \right) \right]^{-1} < 1$$

Thus it follows that the matrix $\mathbf{I}-(\mathbf{I}+\mathbf{A})^{-1}$ is column diagonally dominant with positive diagonal elements. **Q.E.D.**

Corollary 1. If $\mathbf{A}=[a_{ij}]$ is a $n\times n$ row (resp., column) diagonally dominant matrix with positive diagonal elements and $\mathbf{D}=[d_i]$ is a $n\times n$ positive diagonal matrix, then the matrix $(\mathbf{A}^{-1}+\mathbf{D})^{-1}=(\mathbf{I}+\mathbf{AD})^{-1}\mathbf{A}$ is also row (resp., column) diagonally dominant with positive diagonal elements.

Proof. If $\mathbf{A}$ is a row diagonally dominant matrix with positive diagonal elements and $\mathbf{D}$ is a positive diagonal matrix, then by successive use of Lemma 1, Theorem 4, and again Lemma 1 we have that the matrices $\mathbf{DA}$, $((\mathbf{DA})^{-1}+\mathbf{I})^{-1}$, and $\mathbf{D}^{-1}((\mathbf{DA})^{-1}+\mathbf{I})^{-1}=(\mathbf{A}^{-1}+\mathbf{D})^{-1}$ are also row diagonally dominant with positive diagonal elements.
In a similar manner, if $\mathbf{A}$ is column diagonally dominant with positive diagonal elements, then the matrices $\mathbf{AD}$, $((\mathbf{AD})^{-1}+\mathbf{I})^{-1}$, and $((\mathbf{AD})^{-1}+\mathbf{I})^{-1}\mathbf{D}^{-1}=(\mathbf{A}^{-1}+\mathbf{D})^{-1}$ are also column diagonally dominant with positive diagonal elements. **Q.E.D.**

Theorem 5. If $\mathbf{A}=[a_{ij}]$ is a $n\times n$ – row or column – diagonally dominant matrix with positive diagonal elements, then for the matrix $\mathbf{B}=(\mathbf{I}+\mathbf{A})^{-1}$ it holds $\lim_{N\to\infty}\mathbf{B}^N=\mathbf{0}$.

Proof. It is well known [1] that a sufficient condition to have $\lim_{N\to\infty}\mathbf{B}^N=\mathbf{0}$ is either $\|\mathbf{B}\|_\infty<1$ or $\|\mathbf{B}\|_1<1$, which for $\mathbf{B}=(\mathbf{I}+\mathbf{A})^{-1}$ follows directly from Theorem 1 when $\mathbf{A}$ is – row or column – diagonally dominant with positive diagonal elements.
As an alternative proof, it is also well known [1] that $\lim_{N\to\infty}\mathbf{B}^N=\mathbf{0}$ if and only if $\rho(\mathbf{B})=\max_{1\le k\le n}|\lambda_k(\mathbf{B})|<1$, where $\lambda_k(\mathbf{B})$, $k=1,\ldots,n$ are the – not necessarily distinct – eigenvalues of $\mathbf{B}$ and $\rho(\mathbf{B})$ is the largest of their magnitudes (called the *spectral*

radius of **B**). Also, for $\mathbf{B} = (\mathbf{I} + \mathbf{A})^{-1}$ it is true that $\lambda_k(\mathbf{B}) = \frac{1}{1 + \lambda_k(\mathbf{A})}$, $k = 1,\ldots,n$ [2].

If **A** is – row or column – diagonally dominant with positive diagonal elements, then it follows from Theorem 2 that $\operatorname{Re}\lambda_k(\mathbf{A}) > 0$, $\forall k = 1,\ldots,n$. This gives:

$$\left|\lambda_k(\mathbf{B})\right| = \frac{1}{\left|1 + \lambda_k(\mathbf{A})\right|} = \frac{1}{\sqrt{\left(1 + \operatorname{Re}\lambda_k(\mathbf{A})\right)^2 + \left(\operatorname{Im}\lambda_k(\mathbf{A})\right)^2}} < 1, \ \forall k = 1,\ldots,n,$$

or finally $\rho(\mathbf{B}) < 1$. **Q.E.D.**

References

[1] Y. Saad, *Iterative Methods for Sparse Linear Systems, 2nd ed.*, SIAM, 2003.

[2] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge, 1990.